THERE ARE EXACTLY ω_1 TOPOLOGICAL TYPES OF LOCALLY FINITE TREES WITH COUNTABLY MANY RAYS

JORGE BRUNO AND PAUL J. SZEPTYCKI

Abstract. Nash-Williams showed that the collection of locally finite trees under the topological minor relation results in a well-quasi-order. As a consequence, Matthiesen proved that the number λ of topological types of locally finite tree must be uncountable. Since $\aleph_1 \leq \lambda \leq \mathfrak{c}$, finding the exact value of λ becomes non-trivial in the absence of the Continuum Hypothesis. In this paper we address this task by showing that $\lambda = \aleph_1$ for locally finite trees with countably many rays. We also partially extend this result to locally finite trees with uncountable many rays.

1. Introduction.

We consider the family $\mathcal{T}$ of locally finite trees with respect to the *topological minor relation* $\leq^\sharp$, where for $T, S \in \mathcal{T}$ we have $T \leq^\sharp S$ if some subdivision of the tree T embeds as a subgraph of S. If two trees are mutually embedded in this way then they are *topologically equivalent* and we write $T \equiv^\sharp S$. Trees that are topologically equivalent are said to be of the same *topological type*. The topological minor relation is a *quasi-order* (i.e., reflexive and transitive) which partially orders the induced equivalence classes. We recall the result of Nash-Williams that the topological minor relation defines a *well-quasi-order* (wqo) of all trees; a quasi-ordered set is wqo if it is well-founded and all antichains are finite. In fact, Nash-Williams proved this result for *rooted trees* since the unrooted case follows directly [8]. A consequence of Nash-Williams result is that chains in $(\mathcal{T}, \leq^\sharp)$ are well-ordered.

Our focus is on the cardinality of the family of equivalence classes $\mathcal{T}/\equiv^\sharp$ and we rely on some basic set-theoretic notions and tools to study this question. Recall that $\aleph_0$ is the cardinality of the natural numbers and we use the set-theoretic notation ω to denote the natural numbers including 0. The *continuum* $\mathfrak{c}$ denotes the cardinality of the set of real numbers and is also denoted $2^{\aleph_0}$, and $\aleph_1$ denotes the cardinality of the first uncountable ordinal ω_1 (we use $\aleph_1$ when discussing cardinality and ω_1 to denote first uncountable ordinal - so ω_1

is an uncountable well-ordered set all of whose proper initial segments are countable). Our set-theoretic notation and terminology is standard and can be found in [3] or [4].

It is relatively straightforward to show that $|\mathcal{T}| = \mathfrak{c}$. A natural question to ask is the following one.

Question 1. *What is the cardinality of the partially ordered set $\mathcal{T}/\equiv^\sharp$?*

This was initially posed by van der Holst [9] and partially answered by Matthiesen [7] by providing an existential proof that $|\mathcal{T}/\equiv^\sharp| \geq \aleph_1$, making explicit use of the well-quasi ordering of $\mathcal{T}$. The first author answered a question from [7] by giving an explicit construction of $\aleph_1$ topological types of locally finite trees [1] – indeed a chain of order type ω_1 was constructed. Let $(X, \leq)$ be a well-founded partially ordered set with only finite antichains. In [5] Kurepa shows that X has cardinality $\geq \kappa$ if, and only if, there exists an ascending chain in X of size κ. In view of this, our strategy to solve Question 1 is by measuring chains in $\mathcal{T}$. It is worth noting that under the assumption of the Continuum Hypothesis (CH) the problem is fully solved. In the absence of any extra set-theoretic assumptions, this problem becomes more interesting. In particular, one is inclined to ask:

Question 2. *Is $|\mathcal{T}/\equiv^\sharp| = \aleph_1$?*

In Section 2.2 we answer Question 2 in the affirmative for the family of locally finite trees with countably many rays. We also provide partial results for families of locally finite trees with uncountably many rays in Section 2.3.

2. Rooted Trees

A **rooted tree** (T, r) is one with a distinguished vertex r called its **root**. Any rooted tree (T, r) generates a partial ordering $\leq_T$ on its set of vertices by establishing that $s \leq_T t$ provided that the unique path from r to t contains s. This is called the **tree order** induced by T. If not such path exists between vertices s, t then we say they are **incompatible**. A tree (T, r) is **perfect** if every vertex $v \in v(T, r)$ has a pair of incompatible $t, s \geq_T v$. A tree (T, r) is a **rooted subtree** (or simply **subtree** when unambiguous from context) of a tree (S, s) if $v(T, r) \subseteq v(S, s)$ and the tree orders coincide. A **subdivision** of a tree (T, r) is any tree obtained subdividing any number of its edges. The **splitting number** of a vertex $v \in (T, r)$, $sp(v)$, is 1 less than the number of edges incident to v. An injective map $\phi : v(T, r) \to v(S, s)$ is an **embedding** if ϕ can be extended to an isomorphism between a subdivision of (T, r) and the smallest subtree (S', s') of (S, s) containing all vertices in $\phi(v(T,r))$, and furthermore, the path between r and s in (S,s) contains no vertex of $(S,s')'$ other than $\phi(r)$. If there exists an embedding $\phi : v(T,r) \to v(S,s)$ then we say that (T,r) is a **rooted topological minor** of (S,s) we write $(T,r) \leq^\sharp (S,s)$. If $(T,r) \leq^\sharp (S,s)$ and $(T,r) \geq^\sharp (S,s)$ then we write $(T,r) \equiv^\sharp (S,s)$. One can readily verify that any such embedding preserves the tree-order preserving. We prove the conjecture for rooted locally finite trees first and extend the result to the unrooted case.

For reasons we expand upon in the sequel, to answer Question 2 we focus instead on the collection of **rooted locally finite trees**, $\mathcal{T}_r$. An unrooted tree can give rise to a multitude of rooted trees depending on the choice of the distinguished vertex and, therefore, $|\mathcal{T}_r| \geq |\mathcal{T}|$. A simple counting argument yields $|\mathcal{T}_r| \leq \mathfrak{c}$; thus we obtain $|\mathcal{T}_r| = |\mathcal{T}|$. As noted previously, if given unrooted trees T, T' with $(T,r) \leq^\sharp (T',r')$ for vertices $r \in v(T)$ and $r' \in v(T')$ then clearly $T \leq^\sharp T'$. It is much simpler to deal with rooted trees than their unrooted versions. In fact, Nash-Williams proved his main result for all rooted trees in [8]. For our purposes we note that $\mathfrak{c} \geq |\mathcal{T}_r/\equiv^\sharp| \geq |\mathcal{T}/\equiv^\sharp| \geq \omega_1$ and thus a positive answer to Question 2 for $\mathcal{T}_r$ implies a positive answer for $\mathcal{T}$. Henceforth, we sometimes refer to rooted trees without making explicit reference to their roots - i.e., a rooted tree (T,r) will sometimes be simply referred to as T.

The tree on ω rooted at 0 and determined by adding one edge between successive natural numbers (i.e., $n+1$ is connected to n and $n+2$ by an edge) will be refered to as the **ray**. Given a rooted tree (T,r) we interpret any subtree isomorphic to the ray as a function $R : \omega \to v(T)$ with $R(n) <_T R(n+1)$ with $R(n)$ incident to $R(n+1)$. A ray R in a rooted tree (T,r) is **maximal** if $r = R(0)$. We use the notation $R_n^\uparrow$ to denote the induced subtree of T generated from all vertices $v \in v(T)$ for which $v \geq_T R(n)$ and $v \not\geq_T R(m)$ for all $m > n$ (see Figure 2).

In the following sections it becomes necessary to forge trees from smaller ones. For convenience, from now on when joining a tree (T,r) to a vertex v of a tree (S,s) we write **glue** when r and the vertex v are identified as a single vertex, and the resulting tree is rooted on s; preserving $(T,r)'s$ and $(S,s)'s$ tree order. **Attach** refers to the use of an edge e for joining v with r and the resulting tree is rooted on s; preserving $(T,r)'s$ and $(S,s)'s$ tree order. See Figure 2 below.

2.1. **Trees with a single maximal ray.** Define $\mathcal{G}$ to be the collection of all rooted finite trees; there are countably many such trees. Consider $\mathcal{G}^\omega$ - the set of all functions $f : \omega \to \mathcal{G}$ - and for each such function let T_f be the tree obtained by gluing a copy of $f(n)$ to $R(n)$; $R_n^\uparrow = f(n)$. The

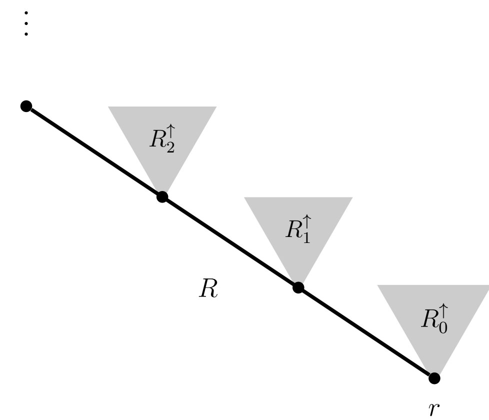

FIGURE 1. Tree (T, r) with a ray R and full subtrees $R_n^{\uparrow}$.

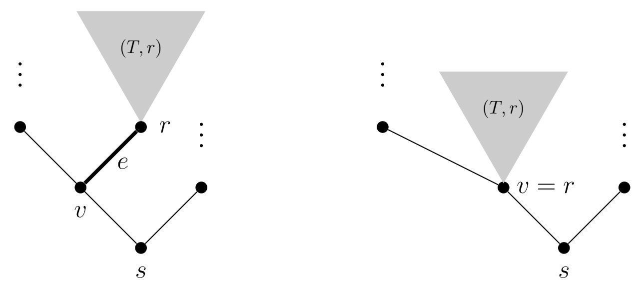

FIGURE 2. Attaching and gluing, respectively.

subsequent lemma makes use of the fact that a quasi-ordered set is wqo precisely when any sequence contains a strictly increasing subsequence [6].

Lemma 1. *Given $f, g \in \mathcal{G}^{\omega}$ the following are equivalent:*

(1) $T_f \leq^{\sharp} T_g$
(2) *There is a strictly increasing sequence $(k_n : n \in \omega)$ such that for all n, $f(n) \leq^{\sharp} g(k_n)$.*

Proof. (2)⇒(1) This follows immediately since we can subdivide the ray of T_f as we see fit in order to suit the sequence $(k_n : n \in \omega)$ for which $f(n) \leq^{\sharp} g(k_n)$.

(1)⇒(2) This follows since a subdivision of the only ray in T_f must be mapped entirely within the only ray in T_g. Indeed, the tree order must be preserved. This leaves the trees attached to each vertex in T_f as topological minors of a sequence of trees in T_g attached to some sequence $(k_n : n \in \omega)$ of vertices in T_g. □

In view of the above, we can interpret $\mathcal{G}^\omega$ as the collection of all rooted locally finite trees containing a single maximal ray. The ordering on $\mathcal{G}^\omega$ is the one permeated from the topological minor relation of the trees that functions from $\mathcal{G}^\omega$ generate. That is, $f \leq g$ in $\mathcal{G}^\omega$ if, and only if, $T_f \leq^\sharp T_g$. Moreover, $[T_g] = \{T_f : f \in [g]\}$ and the ordering $\leq$ on $\mathcal{G}^\omega$ is a wqo. Our first objective is to prove following result.

Theorem 2. *The set $\{[g] : g \in \mathcal{G}^\omega\}$ of equivalence classes is countable.*

By Kurepa's Theorem, it suffices to prove all chains in $\mathcal{G}^\omega$ are countable. In order to do so, we consider a more general setting and obtain Theorem 2 as a special case.

Let $(P, \leq_P)$ be an arbitrary wqo and put $P/\equiv$ as the set of equivalence classes of P with respect to the equivalence relation $S \equiv T$ in P if $S \leq_P T$ and $T \leq_P S$. Next, define a relation $\leq^*$ on P^ω as above: $f \leq^* g$ if there is a strictly increasing sequence (k_n) such that $f(n) \leq_P g(k_n)$. As above, for $f, g \in P^\omega$ define:

- $f \equiv^* g$ if $f \leq^* g$ and $g \leq^* f$; and
- $f <^* g$ if $f \leq^* g$ and $f \not\geq^* g$.

It follows that $\leq^*$ induces a partial ordering of the equivalence classes $P^\omega/\equiv^*$.

Theorem 3. *Assume that the set of equivalence classes $P/\equiv$ is countable. It follows that there are no strictly increasing chains of length $\geq \omega_1$ in $P^\omega/\equiv^*$.*

Proof. Suppose that $\mathcal{F} : \omega_1 \to P^\omega/\equiv^*$ is order preserving (i.e., $\mathcal{F}(\beta) \leq^* \mathcal{F}(\gamma)$ for all $\beta < \gamma$) and for each $\beta \in \omega_1$ put $\mathcal{F}(\beta) = f_\beta$. Recall that a subset A of a partially ordered set $(Q, \leq_Q)$ is said to be cofinal (or dominating) if for each $q \in Q$ there is $a \in A$ such that $q \leq_Q a$.

Definition 4. *We say that $T \in P/\equiv$ is cofinally in the range of $\mathcal{F}$ if for any $n \in \omega$ there are $\beta_n \in \omega_1$ and $m_n > n$ such that $T \leq_P f_{\beta_n}(m_n)$.*

We also define

$$\text{cfrange}(\mathcal{F}) = \{T : T \text{ is cofinally in the range of } \mathcal{F}\}.$$

Lemma 5. *There is an α such that cfrange$(\mathcal{F})$ = cfrange$(\mathcal{F} \restriction \alpha)$.*

Proof. Let $T \in \text{cfrange}(\mathcal{F})$. By definition, for each $n \in \omega$ we can find an α_n and $m_n > n$ with $T \leq_P f_{\alpha_n}(m_n)$. This means that

$$\alpha_T = \sup\{\alpha_n \mid n \in \omega\} < \omega_1$$

since $cof(\omega_1) = \omega_1 > \omega$. Because cfrange($\mathcal{F}$) is countable, another supremum argument forges the desired α with cfrange($\mathcal{F}$) = cfrange($\mathcal{F} \restriction \alpha$) $\square$

From now on, we fix an α as in the above lemma.

Lemma 6. *For all $\beta \geq \alpha$ there is an $n \in \omega$ such that for all $m > n$*

$$f_\beta(m) \in \mathit{cfrange}(\mathcal{F}).$$

Proof. Suppose not. Then there is an infinite increasing sequence $(n_k)_k$ such that for all k, $f_\beta(n_k) = T_k \notin$ cfrange($\mathcal{F}$). Now, since $\leq_P$ is a wqo we have a subsequence $(T_{k_i})_i$ which is increasing with respect to $\leq_P$. By definition, T_{k_0} is in cfrange($\mathcal{F}$). Contradiction. $\square$

Lemma 7. *For any $f \in (\mathit{cfrange}(\mathcal{F}))^\omega$ and for any $\beta \geq \alpha$ we have that $f \leq^* f_\beta$.*

Proof. We fix $f \in$ (cfrange($\mathcal{F}$))$^\omega$ and show that $f \leq^* f_\alpha$. We do so by recursively creating an increasing sequence $(m_n)_n$ with $f(n) \leq_P f_\alpha(m_n)$. Notice that, by design, for any $T \in$ cfrange($\mathcal{F}$) and any $n \in \omega$, since $T \in$ cfrange($\mathcal{F} \restriction \alpha$) we can find an $k_n > n$ and $\beta < \alpha$ with $T \leq_P f_\beta(k_n)$. And since $f_\beta \leq^* f_\alpha$, we can find $m_n > k_n > n$ such that $T \leq_P f_\alpha(m_n)$

Consider $f(0) \in$ cfrange($\mathcal{F}$) = cfrange($\mathcal{F} \restriction \alpha$) and let $m_0 > 0$ such that $f(0) \leq_P f_\alpha(m_0)$. Now, having constructed $m_0 < m_1 < \ldots < m_{k-1}$ so that for all $i < k$ we have $f(i) \leq_P f_\beta(m_i)$, consider $f(k)$. Again, by design, we can find an $m_k > \max\{k, m_{k-1}\}$ with $f(k) \leq_P f_\alpha(m_k)$. This completes the construction of the sequence $(m_n)_n$ which witnesses that $f \leq^* f_\alpha$. $\square$

Now, to complete the proof of Theorem 3, first note that in the construction above, m_0 could have been chosen arbitrarily large. Thus, if $f \in$ (cfrange($\mathcal{F}$))$^\omega$ the sequence $(m_n)_n$ witnessing $f \leq^* f_\alpha$ can be chosen with m_0 as large as we like. To see that there are no strictly increasing ω_1 chains, it suffices to show that there are only countable many distinct equivalence classes among all the $\{[f_\beta] : \beta < \omega_1\}$ and to see this we claim that for all $\beta > \alpha$, f_β is equivalent to some finite modification of f_α. Since $P/\equiv$ is countable, this suffices. Toward this end, fix $\beta > \alpha$ and choose N large enough so that for all $m \geq N$, $f_\beta(m)$ is cofinally in the range of $\mathcal{F}$. We can do this by Lemma 6 above. Consider

$$\hat{f}_\alpha = f_\beta \restriction N \cup f_\alpha \restriction (\omega \setminus N).$$

Then notice that $\hat{f}_\alpha \restriction N = f_\beta \restriction N$, and also that $\hat{f}_\alpha \restriction (\omega \setminus N)$ can be viewed as a function in (cfrange($\mathcal{F}$))$^\omega$. Therefore, by Lemma 7 there is a sequence $(k_m)_{m\geq N}$ witnessing $\leq^*$ which can be chosen so that the initial element $k_N \geq N$. Thus, it follows by extending the sequence so that $k_i = i$ for all $i < n$ that $\hat{f}_\alpha \leq^* f_\beta$.

In the same way we have that $f_\beta \leq^* \hat{f}_\alpha$ and so there are only countably many distinct equivalence classes among the $\{[f_\beta] : \beta \geq \alpha\}$ completing the proof of the theorem. □

The same proof verifies that, in fact, Theorem 3 is true for regular cardinals.

Corollary 8. *Let κ be a regular cardinal, P denote a wqo and assume that $|P/\equiv| = \kappa$. It follows that there are no well-ordered strictly increasing chains in P^ω of cardinality $> \kappa$.*

2.2. **Trees with countably many maximal rays.** For each $n \in \omega$ we let $\mathbf{T}_n$ represent the full n-ary tree. That is, the rooted infinite tree with splitting number n on every vertex. Any perfect tree has $\mathbf{T}_2$ as a topological minor. In the sequel, for convenience of notation, we often drop the reference r to the root of a tree (T, r) when clear from context.

Let S_0 denote the single vertex tree, S_1 be the ray, S_2 be the rooted tree composed of the ray with a copy of S_1 attached at every node, and in general:

- For $\beta = \alpha + 1$: S_β is the rooted ray with a copy of S_α attached at every node.
- For β limit: select a cofinal $\psi_\beta : \omega \to \beta$ and let S_β be comprised of the ray with a copy of $S_{\psi_\beta(n)}$ attached to its n^{th} node.

Denote $\mathcal{S} = \{S_\alpha \mid \alpha < \omega_1\}$. By design, $\alpha < \beta$ implies S_α embeds into S_β.

Remark 9. *Notice that the choice of cofinal function ψ_β is irrelevant in the sense of topological embedability. That is, any two such functions yield topologically equivalent trees. This fact is important in the proof of Lemma 12.*

For a fixed successor $\beta = \alpha + 1 < \omega_1$ (resp. limit $\beta < \omega_1$) we define the **spine** $s(S_\beta)$ of S_β to be the ray where the copies of S_α (resp. $S_{\psi_\beta(n)}$) are attached to, as illustrated above. It is helpful to note that the spine of each S_α is, loosely speaking, its "main" maximal ray. Indeed, notice that any self-embedding of S_2 must map a subdivision of $s(S_2)$ into $s(S_2)$; by design, no subdivision of $s(S_2)$ can be mapped to any branch $s(S_2)^\uparrow_n$.

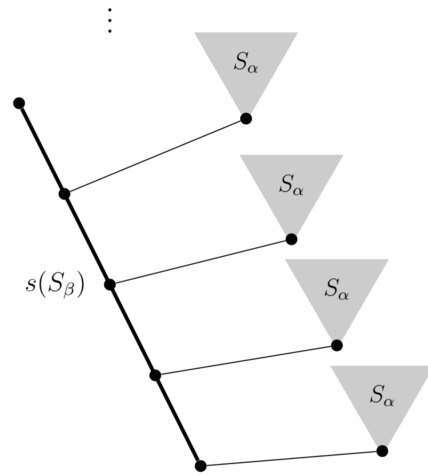

FIGURE 3. The tree S_β for $\beta = \alpha + 1$ with its spine highlighted.

The same is true for S_3, S_4, etc. Since by design $S_n \leq^\sharp S_{n+1}$ and $S_{n+1} \not\leq^\sharp S_n$ we can deduce the same for S_ω. Transfinite induction is employed in [1] to extend the spine argument to all $\alpha < \omega_1$ and yield the following result.

Lemma 10. *The collection $\mathcal{S} = \{S_\alpha \mid \alpha < \omega_1\}$ contains ω_1 topological types of locally finite trees. Moreover for any tree T the following are equivalent:*

(1) *$S_\alpha \leq^\sharp T$ for all $\alpha \in \omega_1$;*
(2) *$\mathbf{T}_2 \leq^\sharp T$; and*
(3) *T has $\mathfrak{c}$ many maximal rays.*

Proof. That (2) and (3) are equivalent is well-known and (2) clearly implies (1). An argument similar to the one given in Lemma 12 below proves that any tree T satisfying (1) contains a perfect subtree. □

In the sequel we employ $\mathcal{S}$ to stratify the collection of locally finite trees with countably many rays.

Definition 11. *Define $o : \mathcal{T}_r \to \omega_1 + 1$ by*

$$o(T) = sup\{\alpha : S_\alpha \leq^\sharp T\}$$

Call this function the **order** *of a tree.*

Let $\mathcal{T}_{\omega_1} \subset \mathcal{T}_r$ denote the collection of all rooted trees with countably many rays. By Lemma 10, $o(T) < \omega_1$ for any $T \in \mathcal{T}_{\omega_1}$: finite trees have order 0, any infinite tree has order at least 1 (by Konig's Lemma any infinite locally finite tree must have a ray), trees with finitely many maximal rays have order one and any tree T with $\mathbf{T}_2 \leq T$ has order ω_1. It is simple to check that given trees $T, S \in \mathcal{T}_{\omega_1}$ with $T \leq^\sharp S$ it follows that $o(T) \leq o(S)$. Hence, considering ω_1 and $\mathcal{T}_{\omega_1}$ as categories, the order assignment is functorial and the assignment $\alpha \mapsto S_\alpha$ defines a left adjoint to it.

Lemma 12. *For any $T \in \mathcal{T}_{\omega_1}$ and $\alpha \in \omega_1$, the following are equivalent:*

(1) *$o(T) = \alpha$;*

(2) *$S_\alpha \leq^\sharp T$ and $S_\gamma \not\leq^\sharp T$ for all other $\gamma > \alpha$; and*

(3) *there are only finitely many maximal rays R of T so that given any embedding $\psi : v(S_\alpha) \to v(T)$ witnessing $S_\alpha \leq^\sharp T$ we have $\psi[v(s(S_\alpha))] \subseteq v(R)$.*

Proof. $[(1) \Leftrightarrow (2)]$ Necessity follows from the definition of o and to prove sufficiency we must only show that $o(T) = \alpha \Rightarrow S_\alpha \leq^\sharp T$. The claim is immediate for a successor α, therefore we focus on the limit case. Let r denote the root of T and assume that α is a limit ordinal. For every $\gamma < \alpha$ put $\phi_\gamma : v(S_\gamma) \to v(T)$ as the embedding that witnesses $S_\gamma \leq^\sharp T$. Denote S'_γ as the subtree of T spanned by ϕ_γ. Define R_γ to be the maximal ray in T into which ϕ_γ maps $v(s(S_\gamma))$ cofinally. Start from r and choose an edge e_1 incident on r so that there exists a cofinal in α set A_1 with the property that all rays R_γ with $\gamma \in A_1$ contain e_1. This is possible since T is locally finite and there are infinitely many R_γ. Repeat the above for the next edge; choose an edge e_2 incident on e_1, not incident on r for which there exists set $A_2 \subseteq A_1$ cofinal in α and so that all rays R_γ with $\gamma \in A_2$ contain e_2. The same reasons that allowed us to choose e_1 also apply to e_2. Continue this way and construct the maximal ray $B = e_1 e_2 \ldots$. We claim that $S_\alpha \leq^\sharp T$ is witnessed by an embedding $\phi_\alpha : v(S_\alpha) \to v(T)$ that maps $s(S_\alpha)$ into B. Put $A = \bigcap_n A_n$. There are two cases to consider:

CASE 1: A is cofinal in α.

Let $\psi : \omega \to A$ be a cofinal map. Hence, $\psi(\omega)$ is also cofinal in α. For each $n \in \omega$ put $\beta_n = \psi(n)$ and for each β_n pick a $\gamma_n < \beta_n$ with $\gamma_n \geq \beta_{n-1}$. The sequence $(\gamma_n)_n$ is also cofinal in α. Recall that, by construction, since $\gamma_n < \beta_n$ then there exists an $m_n \in \omega$ with $S_{\gamma_n} \leq^\sharp s(S_{\beta_n})^\uparrow_{m_n}$ (i.e., the full subtree of S_{β_n} whose root is the m_n^{th} vertex along $s(S_{\beta_n})$). In fact, recall that we can choose m_n to be arbitrarily large.

Start with 0 and let $m_0 \in \omega$ with $S_{\gamma_0} \leq^\sharp s(S_{\beta_0})_{m_0}$. This yields a $k_0 \geq m_0$ with $S_{\gamma_0} \leq^\sharp B^\uparrow_{k_0}$ (i.e., the full subtree of T whose root is the k_0^{th} vertex along B). Indeed, the injection $\phi_{\beta_0} : v(S_{\beta_0}) \to v(T)$ that witnesses $S_{\beta_0} \leq^\sharp T$ maps all vertices of $s(S_{\beta_0})$ to vertices of B. Now, assume that we constructed a sequence $k_0 < k_1 < \ldots < k_{t-1}$ with $S_{\gamma_i} \leq^\sharp B^\uparrow_{k_i}$ for each $i \leq t-1$. We can thus find an $m_t \in \omega$ with $S_{\gamma_t} \leq^\sharp s(S_{\beta_t})^\uparrow_{m_t}$ large enough so that $\phi(m_t) > k_{t-1}$. Putting $k_t = \phi(m_t)$ yields $S_{\gamma_t} \leq^\sharp B^\uparrow_{k_t}$ and extends the sequence $k_0 < k_1 < \ldots < k_{t-1}$. Since $(\gamma_n)_n$ is cofinal in α by Remark 9 the tree composed of $B \cup \{B^\uparrow_{k_n} : n \in \omega\}$ is topologically equivalent to S_α.

CASE 2: A is not cofinal in α.

Construct $\{\gamma_n : n \in \omega\}$ increasing and cofinal in α and $t \in \omega^\omega$ recursively so that $\gamma_n \in A_{t(n)} \setminus A_{t(n+1)}$ for each $n \in \omega$. For each n let k_n be the last vertex along B that ϕ_{γ_n} hits with a vertex from $s(S_{\gamma_n})$. It is evident that $S_{\gamma_n} \leq^\sharp B^\uparrow_{k_n}$. Distill from $(k_n)_n$ a monotone increasing subsequence $(y_n)_n$; this is possible since A is not cofinal in α. The tree $B \cup \{B^\uparrow_{y_n} : n \in \omega\}$ is topologically equivalent to S_α.

[(1) $\Leftrightarrow$ (3)] The same cofinal argument employed above shows that if there is an infinite number of copies of S_α in T so that the copies of their respective spines encompass more than a finite number of maximal rays in T, then one can construct a copy of $S_{\alpha+1}$ in T. This a clear contradiction of $o(T) = \alpha$. $\square$

In view of Lemma 12 (3) for any $T \in \mathcal{T}_{\omega_1}$ with $o(T) = \alpha$ we write $\mathcal{B}(T)$ to denote the collection of all maximal rays R in T with the property that there exists an embedding $v(S_\alpha) \to v(T)$ witnessing $S_\alpha \leq^\sharp T$ that maps all vertices of $s(S_\alpha)$ to vertices of R. We call any such maximal ray $R \in \mathcal{B}(T)$ **principal**.

Lemma 13. *For $T \in \mathcal{T}_{\omega_1}$ it follows that:*

(1) *$\mathcal{B}(T)$ is non-empty and finite;*

(2) *if $S \in \mathcal{T}_{\omega_1}$ with $o(T) = o(S)$ and $T \leq^\sharp S$ with an embedding $\phi : v(T) \to v(S)$ witnessing this, then for any $R \in \mathcal{B}(T)$ there exists a unique $B \in \mathcal{B}(S)$ with $\phi(v(R)) \subseteq \phi(v(B))$; and*

(3) *for any $R \in \mathcal{B}(T)$ and any $n \in \omega$ with $o(R^\uparrow_n) = o(T)$ we have $|\mathcal{B}(R^\uparrow_n)| \leq |\mathcal{B}(T)|$.*

Proof. (1) follows directly from the previous lemma and (3) follows from (2).

(2) Notice that the maximal ray B containing $\phi(v(R))$ must belong to $\mathcal{B}(S)$. Indeed, if $\psi : v(S_\alpha) \to v(T)$ witnesses $S_\alpha \leq^\sharp T$ mapping all

vertices of $s(S_\alpha)$ into $v(R)$ then $\phi \circ \psi$ witnesses $S_\alpha \leq^\sharp S$ mapping all vertices of $s(S_\alpha)$ into $v(R)$. Thus $B \in \mathcal{B}(S)$. $\square$

For $1 \leq n \in \omega$ and $1 \leq \alpha \in \omega_1$ we denote $\mathcal{T}_\alpha^n$ denote the collection of all $T \in \mathcal{T}_{\omega_1}$ with $o(T) = \alpha$ and $|\mathcal{B}(T)| = n$. Therefore, to show that $|\mathcal{T}_{\omega_1}/\equiv^\sharp| = \omega_1$ it suffices to prove that $|\mathcal{T}_\alpha^n/\equiv^\sharp| = \omega$ for each $\alpha \in \omega_1$ and $n \in \omega$.

Theorem 14. *For any $n \in \omega$ and any $\alpha \in \omega_1$ any strictly increasing chain of trees from $\mathcal{T}_\alpha^n$ is countable.*

Proof. We induct on $\omega_1 \setminus \{0\} \times \omega \setminus \{0\}$. The $(1, 1)$ case was dealt with in Theorem 3.

Case $(\alpha, 1)$: assume that $\beta + 1 = \alpha$ and that the claim holds for all (γ, n) with $\gamma \leq \beta$. Observe that any tree in $T \in \mathcal{T}_\alpha^1$ can be identified with a function

$$f_T \in \left(\bigcup\{\mathcal{T}_\gamma^n \mid n \in \omega, \gamma \leq \beta\}\right)^\omega.$$

Therefore, for any $S \in \mathcal{T}_\alpha^1$ we have that $S \leq^\sharp T$ if and only if $f_S \leq^* f_T$ since by Lemma 13 (b) it must be that the only principal ray of S must be mapped to the only principal ray of T. Using our inductive hypothesis that there are only countably many topological types in each $\mathcal{T}_\gamma^n$, $n \in \omega$ and $\gamma \leq \beta$, and applying Theorem 3 we see chains in $T \in \mathcal{T}_{\alpha+1}^1$ consist of at most countably many topological types. A similar argument also establishes that if the claim holds true for $(\alpha, 1)$ when α is a limit ordinal.

Case (α, N) with $N > 1$: fix an $\alpha \in \omega_1 \setminus \{0\}$ and $N > 1$ assume that the result holds for all $(\beta, n) < (\alpha, N)$ in the lexicographic ordering of $\omega_1 \setminus \{0\} \times \omega \setminus \{0\}$. Next we show it holds true for (α, N). Let $\mathcal{C} = \{T_\beta \mid \beta \in \omega_1\}$ be a chain of trees from $\mathcal{T}_\alpha^N$ and for each pair $\beta < \gamma \in \omega_1 \setminus \{0\}$ fix an embedding $\phi_\gamma^\beta : v(T_\beta) \to v(T_\gamma)$ that witnesses $T_\beta \leq^\sharp T_\gamma$. We now employ these embeddings to recursively select for each $\beta \in \omega_1 \setminus \{0\}$ a ray $R_\beta \in \mathcal{B}(T_\beta)$ as follows.

Begin by choosing any ray $R_0 \in \mathcal{B}(C_0)$ and put $R_1 \in \mathcal{B}(C_1)$ as the only ray that contains $\phi_1^0(v(R_0))$. In general, if $\gamma = \eta + 1$ then put $R_\gamma \in \mathcal{B}(T_\gamma)$ as the only ray containing $\phi_\eta^\gamma[v(R_\eta)]$. At a limit stages γ let $B_\gamma \in \mathcal{B}(T_\gamma)$ so that for a cofinal $\overline{\gamma} \subseteq \gamma$ we have $\phi_\beta^\gamma[v(B_\beta)] \subseteq B_\gamma$ for all $\beta \in \overline{\gamma}$. A simple use of transfinite induction and function composition illustrates that the selection of the above distinguished principal rays allows for the following.

Lemma 15. *For all $\beta < \gamma < \omega_1$ there is an embedding $v(T_\beta) \to v(T_\gamma)$ witnessing $T_\beta \leq^\sharp T_\gamma$ such that $v(R_\beta)$ is mapped into $v(R_\gamma)$*

Notice that the embedding $v(T_\beta) \to v(T_\gamma)$ witnessing $T_\beta \leq^\sharp T_\gamma$ guaranteed by the previous lemma might be different to the originally chosen ϕ^β_γ. This is, however, irrelevant. What is important to distill from Lemma 15 is that any tree $T_\gamma \in \mathcal{C}$ can now be identified with a unique function $f_\gamma \in P^\omega$ where

$$P = \bigcup\{\mathcal{T}^n_\beta : \beta < \alpha, n \in \omega\} \cup \bigcup_{m<N} \mathcal{T}^m_\alpha,$$

in such a way so that for $\gamma < \beta < \omega_1$ then $f_\gamma \leq^* f_\beta$. Indeed, by Lemma 13 (c) and the definition of order it follows that for each $n \in \omega$ the tree $(R_\beta)^\uparrow_n \in P$. In addition, the embeddings guaranteed by Lemma 15 provides the cohesion; for $S, T \in \mathcal{C}$ we have $T \leq^\sharp S$ if, and only if, $f_T \leq^* f_S$ as in Theorem 3. Since P is countable we employ Theorem 3 once more to deduce that the chain $\{T_\beta : \beta < \omega_1\}$ is eventually constant. This complete the proof. □

Corollary 16. *There are exactly ω_1 topological types of locally finite trees with countably many rays.*

Any tree with splitting number ≤ 2 is a topological minor of $\mathbf{T}_2$ and since $\mathbf{T}_2$ is a topological minor of any perfect tree, we have the following result.

Corollary 17. *Any strictly increasing chain of trees where each vertex on every tree has splitting number ≤ 2 is of length $\leq \omega_1$.*

2.3. **Trees with uncountably many maximal rays.** The answer to Question 1 for chains of trees with uncountably many branches remains open. However, in this section we provide several partial answers and in order to do so we continue working within $\mathcal{T}_r$. Note that the previous corollary establishes that chains of trees with splitting number ≥ 2 on every vertex are of order type at most $\omega_1 + 1$. In this section we extend the result to collections of trees with a *restricted* number of vertices with splitting number ≥ 3. Therefore, we begin by investigating the length of chains of trees with finitely many vertices of splitting number 3 and all other vertices of splitting number ≤ 2.

Lemma 18. *Let $\mathcal{C}$ be a strictly increasing chain in $\mathcal{T}_r$. If there exists a tree $T \leq^\sharp \mathbf{T}_3$ with exactly one vertex of splitting number* 3 *so that $S \leq^\sharp T$ for all $S \in \mathcal{C}$ then $|\mathcal{C}| \leq \omega_1$.*

Proof. Let T be the tree that results from attaching an extra copy of T_2 to the root of T_2 itself; all vertices of T have splitting number 2 with the exception of its root, which has splitting number 3. Let $\mathcal{C} = \{C_\alpha \mid \alpha \in \kappa\}$ be a chain where $C_\alpha \leq^\sharp T$ for all $\alpha \in \kappa$. Notice that if any $C_\alpha \in \mathcal{C}$ contains a vertex of splitting number 3 then it must be its root, by virtue of $C_\alpha \leq^\sharp T$. If $\mathcal{C}$ contains no C_α with splitting number 3 on its root then by Corollary 17 it follows that $|\mathcal{C}| \leq \omega_1$ and we are done. Otherwise, we focus on the length of the upper part of $\mathcal{C}$ composed of trees whose root splitting number is 3. Denote this new chain by $\mathcal{D} = \{D_\alpha : 0 \leq \alpha \leq \kappa\}$. We show that $|\kappa| \leq \omega_1$.

For each $D_\alpha \in \mathcal{D}$ let r_α denote its root and T's root by r. As already established, for $\alpha < \beta$ any embedding $v(D_\alpha) \to v(D_\beta)$ must send $r_\alpha \mapsto r_\beta$. Because each r_α has splitting number 3 we can think of each D_α as 3 trees $D^0_\alpha, D^1_\alpha, D^2_\alpha$ attached to r_α. For each tree $D_\alpha \in \mathcal{D}$ we refer to such subtrees as the *leaves* of D_α. In particular, for $\alpha < \beta$ any embedding $v(D_\alpha) \to v(D_\beta)$ witnessing $D_\alpha \leq^\sharp D_\beta$ must map all of $v(D^i_\alpha)$ into a unique $v(D^{j_i}_\beta)$ where $i = 0, 1, 2$ and $j_i = 0, 1, 2$ so that $j_l = j_k$ precisely when $l = k$. That is, by virtue of having to map roots to roots any such embedding creates a one-to-one correspondence between leaves of D_α and leaves of D_β.

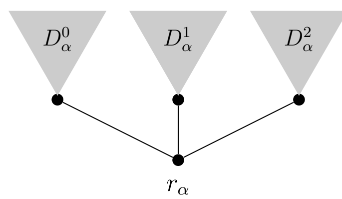

FIGURE 4. The tree D_α.

For each pair $\alpha < \beta$ fix an embedding $\psi^\alpha_\beta : v(D_\alpha) \to v(D_\beta)$ witnessing $D_\alpha \leq^\sharp D_\beta$. Starting with $n = 0$ put L_0 as the leaf D^0_0 of D_0 and let L_1 be the leaf D^i_1 with $\psi^0_1[v(L_0)] \subseteq v(L_1)$. For $\gamma \in \kappa$ a successor ordinal $\gamma = \eta + 1$ set L_γ be the leaf of D_γ for which $\psi^\eta_\gamma[v(L_\eta)] \subseteq v(L_\gamma)$. For any limit ordinal $\gamma \in \kappa$, let L_γ represent a leaf D^i_ω for which there exists a cofinal $A_\gamma \subset \gamma$ with $\psi^\alpha_\gamma[v(L_\alpha)] \subseteq v(L_\gamma)$ for all $\alpha \in A_\gamma$. This process selects a unique leaf L_α from each D_α where $\alpha < \beta$ implies $L_\alpha \leq^\sharp L_\beta$. Moreover, since each leaf $L_\alpha \leq^\sharp T_2$ by Corollary 17 the chain $\{L_\alpha \mid \alpha \in \kappa\}$ plateaus at a stage $< \omega_2$; there exists an $\eta \in \omega_2$ with $L_\alpha \equiv^\sharp L_\beta$ for all $\alpha, \beta \geq \eta$.

Just as we selected the first leaf from each D_α we repeat for the second leaf. Put M_0 as D^1_0 and let M_1 be the leaf D^i_1 with $\psi^0_1[v(M_0)] \subseteq v(M_1)$. For $\gamma \in \kappa$ a successor ordinal $\gamma = \eta + 1$ set M_γ be the leaf of D_γ for which $\psi^\eta_\gamma[v(M_\eta)] \subseteq v(M_\gamma)$. For any limit ordinal $\gamma \in \kappa$, consider the cofinal set $A_\gamma \subset \gamma$ as employed when selecting L_γ and let M_γ represent a leaf D^i_ω for which there exists a cofinal $B_\gamma \subseteq A_\gamma$ with $\psi^\alpha_\gamma[v(M_\alpha)] \subseteq v(M_\gamma)$ for all $\alpha \in B_\gamma$.

Lemma 19. *For each $\gamma \in \kappa$, M_γ and L_γ identify different leaves of D_γ.*

Proof. The successor stage is trivial and we focus on the limit stage. Assume that γ is a limit ordinal with cofinal $A_\gamma \subseteq \gamma$ employed to select L_γ. By the inductive hypothesis we get that for each $\beta < \gamma$, L_β and M_β identify different leaves on each D_β. Hence for each D_β with $\beta \in A_\gamma$ the embedding ψ^β_γ maps $v(M_\beta)$ into a leaf of D_γ distinct to from L_γ. There are only two leaves left in D_γ distinct from L_γ and, therefore, our claim that there must exist another leaf M_γ of D_γ so that for a cofinal $B_\gamma \subseteq A_\gamma$ we get $\psi[v(M_\beta)] \subseteq v(M_\gamma)$ for all $\beta \in B_\gamma$. $\square$

As with our first choice of leaves, $M_\alpha \leq^\sharp M_\beta$ for $\alpha \leq \beta$ and the chain $\{M_\alpha \mid \alpha \in \kappa\}$ plateaus at a stage $< \omega_2$; there exists an $\eta \in \omega_2$ with $M_\alpha \equiv^\sharp M_\beta$ for all $\alpha, \beta \geq \eta$. To finish this stage of the proof, we can identify the third leaf N_α of each D_α distinct from L_α and M_α starting from D_0 to obtain the last chain $\{N_\alpha \mid \alpha \in \kappa\}$ which eventually plateaus at a stage $< \omega_2$. It is then straightforward to conclude that the chain $\mathcal{D}$ plateaus at a stage $< \omega_2$.

A moment's thought verifies that there is nothing special about $T's$ root being the only vertex of splitting number 3 in the above construction. Indeed, assume that v is T's only vertex of splitting number 3 (not necessarily T's root) and $\mathcal{C} = \{T_\alpha \mid \alpha \in \kappa\}$ a chain with $T_\alpha \leq^\sharp T$. The embeddings $v(T_\alpha) \to v(T)$ witnessing $T_\alpha \leq^\sharp T$ allow for only 1 vertex of splitting number 3 for each T_α. Therefore, we then interpret each T_α as being composed of 4 disjoint trees that are topological minors of $\mathbf{T}_2$; three (D^i_α with $i = 0, 1, 2$) attached to its only vertex of splitting number 3 and the last one (D^3_α) encompassing what is left as in Figure 5.

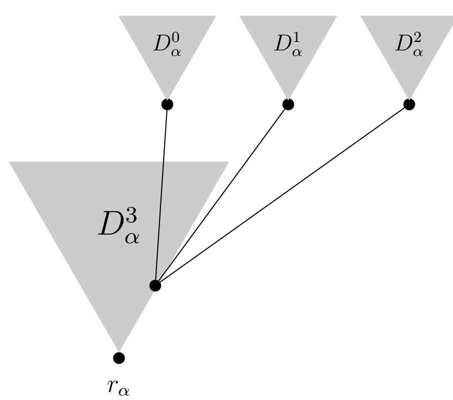

FIGURE 5. The tree T_α.

As before, the embeddings $\psi^\alpha_\beta : v(T_\alpha) \to v(T_\beta)$ witnessing $T_\alpha \leq^\sharp T_\beta$ with $\alpha < \beta$ must send $\psi[v(D^i_\alpha)] \subseteq v(D^{j_i}_\beta)$ with $i = 0, 1, 2$, $j_i = 0, 1, 2$ with $j_l = j_k$ precisely when $l = k$. Hence, $\psi[v(D^3_\alpha)] \subseteq v(D^3_\beta)$. This allows us to apply the same logic as with the root case and obtain our result. □

It is worth noting that the order type of $\mathcal{C}$ can be larger than ω_1. Regardless, the cardinality of $\mathcal{C}$ can not exceed ω_1. Simple induction shows that Lemma 18 holds even for a tree $T \leq^\sharp \mathbf{T}_3$ with any finite number of vertices with splitting number 3. In fact, another use of simple induction verifies the same is true when the finite number of vertices has splitting number ≥ 3.

Corollary 20. *Let $\mathcal{C}$ be a strictly increasing chain of locally finite trees. If there exists a tree T with at most finitely many vertices of splitting number > 2 so that $S \leq^\sharp T$ for all $S \in \mathcal{C}$ then $|\mathcal{C}| \leq \omega_1$.*

When there is no finite bound to the number of nodes with splitting number ≥ 3 in a chain of locally finite trees the scenario becomes more complex.

Example 21. *Consider the case of a chain $\mathcal{C}$ for which there exists a tree T so that $S \leq^\sharp T$ for all $S \in \mathcal{C}$ and where T has infinitely many vertices of splitting number ≥ 3 and all such vertices belong to the same maximal ray R of T. In view of Corollary 20 let us assume that each tree S also has an infinite number of vertices of splitting number ≥ 3. Again, by virtue of $S \leq^\sharp T$ for all $S \in \mathcal{C}$ it must be that each S has a distinguished ray R_S to which all vertices of splitting number ≥ 3 belong. This creates a scenario much like the one encountered in Theorem 14; indeed, each $S \in \mathcal{C}$ can be identified with a function*

$$f_S \in \{B \in \mathcal{T}_r \mid B \leq \mathbf{T}_2\}^\omega$$

where for each $n \in \omega$ the tree $f_S(n)$ is glued to R_S's n^{th} vertex. We employ Corollary 8 and Corollary 17 to conclude that $\mathcal{C}$ is composed of at most ω_1 topological types of trees.

The scenario illustrated above would yield the same result if we replace T with another tree with finitely many maximal rays spanned by infinitely many vertices of splitting number ≥ 3. We won't prove this claim. Instead we prove something stronger. In fact, our approach to dealing with chains of trees with infinitely many vertices of splitting number ≥ 3 is to *map out* maximal rays that witness infinitely vertices of splitting number ≥ 3. Next, we formalise this notion.

Definition 22. *For any $T \in \mathcal{T}_r$ define $T^\star$ as the subtree of T spanned by all maximal rays R of T so that for each $n \in \omega$ there exists an $m \geq n$ and a vertex $v \geq_T R(m)$ with $sp(v) \geq 3$.*

One can readily verify that for trees $S \leq^\sharp T$ it follows $S^\star \leq^\sharp T^\star$. Moreover, if $o(S^\star) = o(T^\star)$ then for any embedding $\psi : v(S) \to v(T)$ witnessing $S \leq^\sharp T$ and any $R \in \mathcal{B}(S^\star)$ there must exist a $B \in \mathcal{B}(T^\star)$ so that $\psi[v(R)] \subseteq v(B)$.

Lemma 23. *Given a strictly increasing chain $\mathcal{C}$ so that for each $T \in \mathcal{C}$ we have $o(T^\star) \in \omega_1$ it follows that $|\mathcal{C}| \leq \omega_1$.*

Proof. Let $\mathcal{C}$ be as required. We show that for any $n \in \omega$ and $\alpha \in \omega_1$ there are no more than ω_1 trees $T \in \mathcal{C}$ with $T^\star \in \mathcal{T}^n_\alpha$. By Theorem 14 the rest follows immediately.

The case with $\alpha = 0$ is settled by Corollary 20. Fix an $(N, \alpha) \in \omega \times \omega_1$ so that for any other pair $(\beta, n) < (\alpha, N)$, relative to the lexicographic order on $\omega_1 \times \omega$, the claim holds. Put

$$\mathcal{D} = \{D \in \mathcal{C} \mid D^\star \in \mathcal{T}^N_\alpha\}$$

and for each pair $D_\beta \leq^\sharp D_\gamma$ in $\mathcal{D}$ with $\beta < \gamma$ fix an embedding $\psi^\beta_\gamma : v(D_\beta) \to v(D_\gamma)$ witnessing this. The approach here is the same as the one in Theorem 14. Fix a ray $R_0 \in \mathcal{B}(D^\star_0)$ and recall that there must exist a ray $R_1 \in \mathcal{B}(D^\star_1)$ with $\psi^0_1[v(R_0)] \subseteq v(R_1)$. In general, if $\gamma = \eta + 1$ then put $R_\gamma \in \mathcal{B}(D^\star_\gamma)$ as the only ray containing $\phi^\gamma_\eta[v(R_\eta)]$. At a limit stages γ let $R_\gamma \in \mathcal{B}(D^\star_\gamma)$ so that for a cofinal $\overline{\gamma} \subseteq \gamma$ we have $\phi^\gamma_\beta[v(R_\beta)] \subseteq R_\gamma$ for all $\beta \in \overline{\gamma}$. Putting

$$P = \bigcup\{\mathcal{T}^n_\beta : \beta < \alpha, n \in \omega\} \cup \bigcup_{m<N} \mathcal{T}^m_\alpha.$$

the aforementioned selection of rays allows us to interpret each $D \in \mathcal{D}$ as a function

$$f_D \in \{T \mid T^\star \in P\}^\omega.$$

This completes the proof. □

The above result is the strongest we are able to prove. Concluding remarks and further questions can be found in Section 3

3. Conclusion

In Section 1 we introduced the question of the size of the partially ordered set $\mathcal{T}/\equiv^\sharp$. Even a positive answer to Question 2 might not resolve the following:

Question 3. *What is the supremum of the lengths of strictly increasing chains in $\mathcal{T}/\equiv^\sharp$?*

Notice that a strictly increasing chain C of rooted trees gives rise to another of unrooted trees C' by forgetting the distinguished vertex of each tree in C. Clearly, $|C| \geq |C'|$ and equality is clearly not guaranteed. Hence, the previous question can be asked of $\mathcal{T}_r/\equiv^\sharp$.

Question 4. *What is the supremum of the lengths of strictly increasing chains in $\mathcal{T}_r/\equiv^\sharp$?*

Whilst we are able to construct chains longer than ω_1 - e.g., it is actually possible to construct a chain of inequivalent topological types of locally finite trees of order type $\sum_{n\in\omega+1}\omega_1^n$ by employing trees T with $o(T^\star) < \omega_1$, for all $i \in \omega + 1$ - we do not know the answer to the following questions:

Question 5. *Is there, for each $\alpha < \omega_2$ a chain of inequivalent topological types of locally finite trees of order type α?*

Question 6. *Is it consistent that there is a chain of inequivalent topological types of locally finite trees of order type ω_2 or longer (of course in the presence of the negation of the Continuum Hypothesis)?*

Another surprisingly non-trivial question to ask is inverse of the chain scenario aforementioned:

Question 7. *Does every strictly increasing chain of unrooted trees yield a strictly increasing chain of rooted trees by selecting a distinguished vertex from each tree?*

Our approach to answering Question 2 relies on the use of rooted trees via the observation that $|\mathcal{T}_r/\equiv^\sharp| \geq |\mathcal{T}/\equiv^\sharp|$. A natural question to ask is:

Question 8. *Is $|\mathcal{T}_r/\equiv^\sharp| = |\mathcal{T}/\equiv^\sharp|$?*

Lemma 23 in Section 2.3 poses a limit to the types of chains we are able to measure. Even some of the simplest of cases remain elusive. For example, let $\mathcal{F}$ denote the collection of trees forged as follows: take T_2 and subdivide each edge in two by adding a new vertex. To every new vertex (i.e., every vertex of splitting number 1) glue a finite tree.

Question 9. *What is the size of $|\mathcal{F}|$?*

References

[1] Bruno, J. *A family of ω_1 topological types of locally finite trees*, Discrete Mathematics, v 340, pp. 794-795, 2017.

[2] Diestel, R. **Graph Theory**, Graduate Texts in Mathematics, 2010.

[3] Just, W. and Weese, M, **Discovering Modern Set Theory, I** American Mathematical Society, 1996.

[4] Kunen, K. **Set Theory**, College Publications, 2011.

[5] Kurepa, G *A Propos d'une généralisation de la notion d'ensembles bien ordennés*, Acta Mathematica, v75, 139-150, 1942.

[6] Kruskal, J., *The Theory of Well-quasi-orderings: a frequently discovered concept*, Journal of Combinatorial Theory, Series A, v13, 297-305, 1972.

[7] Matthiesen, L. *There are uncountably many topological types of locally finite trees* Journal of Combinatorial Theory, Series B, v 96, Issue 5, pp 758-760, 2006.

[8] Nash-Williams, C. A., *On well-quasi-ordering infinite trees*, Proc. Camb. Phil. Soc. v 61, pp. 697–720, 1965.

[9] van der Holst, H. *Problem posed at the 2005 Graph Theory workshop at Oberwolfach.*